Macroscopic description of monochromatic waves in a rectangular non-staggered porous medium

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Abstract

The study of flow in porous media is a complex problem in which a lot of research has been carried out in the last decades. Physicists, engineers, hydrologists, and mathematicians contribute to understanding the interaction of water flow and porous media. Different models and theories have been proposed for describing flow in porous media, including analytical and numerical methods that have been used to solve such models. Airy's wave theory that involves modelling waves with small amplitudes will be used to describe the motion of waves. In this study, we are going to describe the motion of water waves in porous media by using Airy's small amplitude wave theory. The volume averaging method will be used to obtain the macroscopic governing equations which describe the propagation of water waves in porous media.

Keywords: water wave, volume averaging, porous medium, velocity potential.

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Joram Divene BAKEKOLO, 23 May 2019

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1. Introduction

1.1 Background

The interactions between waves and porous media have been one of the main challenges in coastal engineering since the last century. Many studies and mathematical models have been proposed in order to understand the behaviour of flow inside porous structures. These interactions can be with rocks, breakwaters, soil and more.

Different models have been proposed, for example, (Sollitt and Cross, 1972) proposed a model to study the interaction between waves and porous structures, where a velocity potential is solved from the Laplace equation. (Hwung et al., 1996) conducted experiments to measure the flow of waves in a permeable sandy bed.

Many numerical solutions have been developed in order to solve this problem, particularly by (Rojanakamthorn et al., 1989) who used the mild-slope equation, (Cruz et al., 1997) used a Boussinesq equation, (Huang et al., 2003) used the Navier-Stokes equation, and (van Gent, 1994) used shallow water equations in order to study the motion of waves inside a permeable structure.

During the last decades, work has been done by (Van Gent, 1995), (Karunarathna and Lin, 2006), (Liu et al., 1999), and (Giménez-Curto et al., 2001) on the 2D Reynold-average Navier-Stokes (RANS) model. The volume averaged and Reynolds-averaged Navier-Stokes equation (VARANS), has empowered the study of flow in porous media with accuracy. (Wen et al., 2018) and (Hu et al., 2012) have tried to study the problem in 3D, also based on the Navier-Stokes equation. Several models have been designed to understand wave interactions with porous media. These different types of models can be explained by the increased complexity of the physics in each model.

All recent approaches that have been used to study the behaviour of waves through a porous medium, start with the Navier-Stokes equation. There are two different approaches, one is microscopic and the other is macroscopic. The microscopic method gives a better view of each component and particle in the media, but unfortunately, it is difficult to use in practice. However, the macroscopic approach works by ignoring the internal geometry of the pores and gives the average properties. In our study we will use the macroscopic approach, which gives the mean flow within the porous medium.

A water wave is created by a transfer of energy, commonly from the wind on the surface of water. When wind blows over the surface of water, the disturbance creates a wave crest. The size of a wave depends on different parameters such as the energy transferred by the wind, the distance that the wind has blown over the water, and the duration of the wind. During wave propagation, energy goes from one point to another without transferring material. A water wave can be classified by the form of the wave, the depth of the water, and its origin. Indeed, by the form of the wave, a wave can be classified as progressive or standing. The depth of the water is also one of the parameters that influence the wave's characteristics.

Airy's wave theory, also called small amplitude wave theory, is a simplified wave theory introduced by George Biddell Airy (Goda, 2010). It describes the propagation of linear waves on the surface of water by using the velocity potential, mostly used for shallow water. However, Stokes's wave theory introduced by George Stokes (Svendsen, 2006) describes non-linear surface waves, mostly used for intermediate and deep water. Besides the theories enumerated above, many other theories have been introduced such as the Boussinesq approximation. The Boussinesq approximation has been introduced by Joseph

Boussinesq in 1871 (Darrigol, 2017). It is an approximation theory for non-linear and fairly long waves. The theory requires the elimination of the vertical coordinate in the flow equation.

1.2 Objectives

Among all the approximations introduced, Airy wave theory, Stokes wave theory, and Boussinesq wave theory, we will use only linear wave theory, which is Airy wave theory, for small amplitudes, to be applied to waves in porous media.

The main objective of this study is to derive the macroscopic governing equations and boundary conditions that describe the motion of a wave through a porous medium. The macroscopic equations will be derived by using the volume averaging theory which provides a sound basis to obtain macroscopic equations.

In general, the secondary objectives are:

- Describe the equations govern wave motion in the absence of porous media;
- Present the set of equations that define linear wave motion;
- Present a general discussion of the volume averaging theory which is used to obtain macroscopic equations for flow through porous media;
- Use the volume averaging theory to obtain the set of macroscopic equations governing linear water waves in a porous medium;
- Present an initial attempt to obtain closure models for the governing macroscopic equations.

In this study, the first milestone is to determine the governing equation and boundary conditions that describe water waves with small amplitudes. The second milestone is to introduce the volume averaging method, and then apply the method to the governing equation obtained. The third milestone is to close the volume averaged equations. It is not the intension of this study to solve the microscopic and macroscopic equations, but to compare the microscopic and macroscopic boundary value problems.

2. Small Amplitude Wave Theory

2.1 Governing Equation

In this section, we will determine the governing equation and boundary conditions which describe the motion of water waves . All throughout our study, Airy's wave theory will be the approach that we will use to describe the motion of the water waves. The theory uses the velocity potential to describe the movement of the surface of the water. The velocity potential is defined as a scalar function of space and time, whose derivative with respect to a direction provides the component of the velocity in that same direction. It is preferable to work with the velocity potential instead of other parameters. Finding the velocity potential makes it easy to deduce other parameters such as the flow velocity, and the water level.

Throughout our study ∇ is written in terms of x and y, and it describes the variation in both directions, and we keep z separate. The z-direction is chosen to be positive upwards and negative downwards from the still water level (SWL). The flow occurs from left to right, and the gravitational force is zero in the x and y directions. To use Airy's wave theory we need to make some assumptions in order to simplify the problem and obtain the model needed. Those assumptions are presented below. We assume that:

- ◇ The surface tension of the fluid is negligible;
- ◇ The density of the fluid is constant ($\rho =$ constant);
- ◇ The seabed is rigid, horizontal and impermeable;
- ◇ The pressure at the free surface is uniform and constant;
- ◇ The vorticity is zero.

Figure 2.1 describes the seabed and the wave surface elevation which is measured from the still water level in our study. We define a new vector ∇_t to generalise the variation in all directions x, y and z. The vector ∇_t is related to ∇ by the relation

$$\nabla_t = \nabla + \frac{\partial}{\partial z}\vec{k}.$$

The flow of water in this study is irrotational, from the zero vorticity assumption the velocity potential ϕ exists, and the velocity components u, v, w, in the x, y, z directions can be found by differentiating the velocity potential, such that

$$\vec{V} = \nabla_t \phi,$$

where $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$, $w = \dfrac{\partial \phi}{\partial z}$ and $\vec{V}$ is the velocity of the fluid.

The velocity $\vec{V}$ can be written as the sum of $\nabla\phi$ and the vertical variation. We have in this case

$$\vec{V} = \nabla\phi + \frac{\partial \phi}{\partial z}\vec{k}. \tag{2.1.1}$$

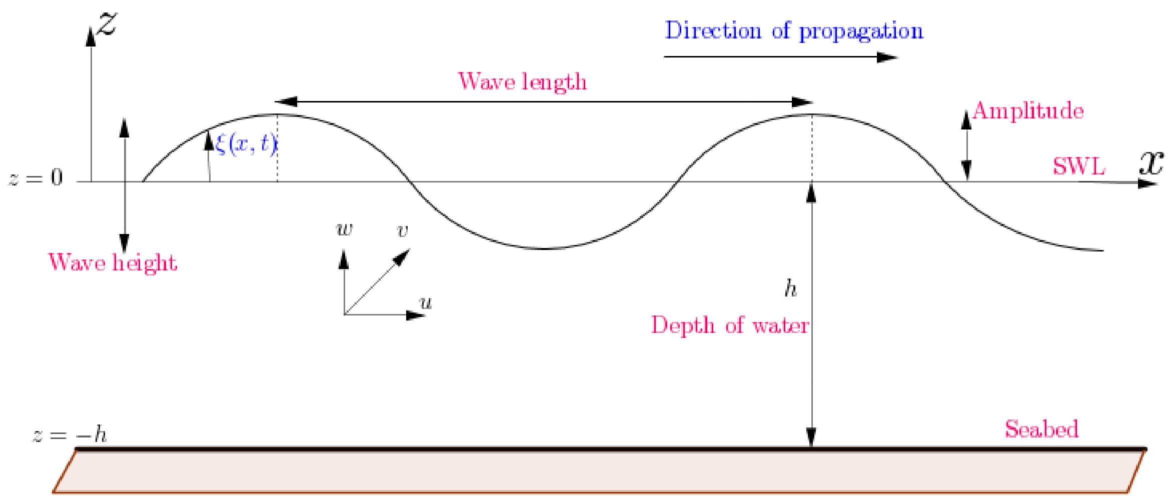


Figure 2.1: Definition sketch for water waves

According to the assumptions that we introduced, the equations that describe the water motion are:

$$\begin{cases} \nabla_t \cdot \vec{V} = 0, & (2.1.2a) \\ \rho \dfrac{d\vec{V}}{dt} = -\nabla_t p + \rho \vec{g}, & (2.1.2b) \\ \nabla_t \times \vec{V} = 0, & (2.1.2c) \end{cases}$$

where ρ is the density of water, $\vec{g}$ the gravitational acceleration, and p the pressure.

Equations (2.1.2a) to (2.1.2c) presented above have physical interpretation as follows:

- ◇ $\nabla_t \cdot \vec{V} = 0$, the density is constant and mass is conserved.
- ◇ $\rho \dfrac{d\vec{V}}{dt} = -\nabla_t p + \rho \vec{g}$, momentum conservation.
- ◇ $\nabla_t \times \vec{V} = \vec{0}$, the flow is irrotational, and therefore there exists a velocity potential, ϕ.

The divergence of $\vec{V}$ defined in Equation (2.1.2a) can be written as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.1.3}$$

Since the velocity $\vec{V}$ is a function of the velocity potential showed in Equation (2.1.1), by rewriting Equation (2.1.3) in terms of the velocity potential, we have

$$0 = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

We obtain an important equation for our study, the Laplace equation in terms of the velocity potential and is given as follows

$$\nabla^2 \phi + \left(\frac{\partial^2 \phi}{\partial z^2}\right) = 0. \tag{2.1.4}$$

Let us consider the second equation in the system presented above, Equation (2.1.2b). Since the gravitational force is zero in the x and y-directions, the simplified Euler's equation can be written in terms of the following scalar components

$$\begin{cases} \dfrac{du}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{dv}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \dfrac{dw}{dt} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g. \end{cases} \tag{2.1.5}$$

We know that the total derivative applied to the velocity vector $\vec{V}$ can be written as follows

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}.$$

The total derivative applied to u, v and w respectively on the left side in Equation (2.1.5) now becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g. \end{cases} \tag{2.1.6}$$

We know that the flow is irrotational, i.e. $\nabla_t \times \vec{V} = \vec{0}$, therefore we have the expression

$$\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} = \vec{0}$$

yielding

$$\begin{cases} \dfrac{\partial w}{\partial y} = \dfrac{\partial v}{\partial z}, \\ \dfrac{\partial u}{\partial z} = \dfrac{\partial w}{\partial x}, \\ \dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y}. \end{cases} \tag{2.1.7}$$

By substituting Equations (2.1.7), the system (2.1.6) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x} + w\dfrac{\partial w}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial w}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial u}{\partial z} + v\dfrac{\partial v}{\partial z} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g. \end{cases} \tag{2.1.8}$$

By substituting $u = \dfrac{\partial \phi}{\partial x}$, $v = \dfrac{\partial \phi}{\partial y}$ and $w = \dfrac{\partial \phi}{\partial z}$ into the partial derivatives with respect to time in the system (2.1.8), it follows that

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t \partial x} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial v}{\partial x} + w\dfrac{\partial w}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}, \\ \dfrac{\partial^2 \phi}{\partial t \partial y} + u\dfrac{\partial u}{\partial y} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial w}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}, \\ \dfrac{\partial^2 \phi}{\partial t \partial z} + u\dfrac{\partial u}{\partial z} + v\dfrac{\partial v}{\partial z} + w\dfrac{\partial w}{\partial z} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g. \end{cases} \tag{2.1.9}$$

The first equation of the system can be written as follows

$$\frac{\partial^2 \phi}{\partial t \partial x} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}w^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

We do the same for the second and the third equations. Integrating this system with respect to x, y, and z respectively, we obtain the equation set below

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{u^2}{2} + \dfrac{v^2}{2} + \dfrac{w^2}{2} = -\dfrac{p}{\rho} + f_1(y,z,t), & \text{(2.1.10a)} \\ \dfrac{\partial \phi}{\partial t} + \dfrac{u^2}{2} + \dfrac{v^2}{2} + \dfrac{w^2}{2} = -\dfrac{p}{\rho} + f_2(x,z,t), & \text{(2.1.10b)} \\ \dfrac{\partial \phi}{\partial t} + \dfrac{u^2}{2} + \dfrac{v^2}{2} + \dfrac{w^2}{2} = -\dfrac{p}{\rho} - gz + f_3(x,y,t), & \text{(2.1.10c)} \end{cases}$$

where $f_1(y,z,t)$, $f_2(x,z,t)$ and $f_3(x,y,t)$ are integration constants.

By subtracting Equation (2.1.10a) from (2.1.10b), we get the equality

$$f_1(y,z,t) = f_2(x,z,t).$$

This equality means that

$$f_1(z,t) = f_2(z,t).$$

Equations (2.1.10a) and (2.1.10b) can be written as follows

$$\frac{\partial \phi}{\partial t} + \frac{u^2}{2} + \frac{v^2}{2} + \frac{w^2}{2} + \frac{p}{\rho} = f_1(z,t).$$

By substituting this equation into Equation (2.1.10c) we get

$$f_1(z,t) = f_3(x,y,t) - gz.$$

This equality is only possible if $f_3(t) = f(t)$, therefore we can reduce equation set (2.1.10a) and (2.1.10c) to

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + w^2) + \frac{p}{\rho} + gz = f(t) \tag{2.1.11}$$

yielding the Bernoulli equation written in terms of the velocity potential ϕ and u, v and w for unsteady flow. We know that a velocity potential is not unique. In Equation (2.1.11), $f(t)$ is a scalar function of time and we may eliminate this integration constant by redefining the velocity potential as

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} - f(t). \tag{2.1.12}$$

Since $u^2 = \left(\frac{\partial \phi}{\partial x}\right)^2$, $v^2 = \left(\frac{\partial \phi}{\partial y}\right)^2$ and $w^2 = \left(\frac{\partial \phi}{\partial z}\right)^2$, Equation (2.1.11) may be written in terms of this new velocity potential, Φ, as follows

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{p}{\rho} + gz = 0. \tag{2.1.13}$$

From our definition of ∇ written in terms of x and y, applied to the velocity potential Φ we obtain the following relation

$$\nabla \Phi = \frac{\partial \Phi}{\partial x}\vec{i} + \frac{\partial \Phi}{\partial y}\vec{j}$$
$$(\nabla \Phi)^2 = \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2.$$

Substituting the above relation into Equation (2.1.13), we obtain

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left((\nabla \Phi)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{p}{\rho} + gz = 0. \tag{2.1.14}$$

This equation is also called Bernoulli's equation for unsteady flow, written in terms of the velocity potential, Φ. It is an important equation for our problem and describes the change of the velocity potential in time from one point to another.

Another important equation obtained for our study is the Laplace equation (2.1.4) which may be rewritten in terms of our newly defined velocity potential as

$$\nabla^2 \Phi + \left(\frac{\partial^2 \Phi}{\partial z^2}\right) = 0. \tag{2.1.15}$$

Equations (2.1.14) and (2.1.15) are linear, second order, homogeneous differential equations, and the velocity potential can be solved analytically or numerically. Supplying a partial differential equation is never sufficient to obtain a unique solution. In addition, we need to define the appropriate boundary conditions and define the region in which Laplace's equation will be valid.

2.2 Boundary Conditions of the Problem

We have two different types of boundary condition, namely

- ◇ kinematic boundary conditions
- ◇ dynamic boundary conditions.

These will be described at the free surface and at the seabed.

2.2.1 Dynamic boundary condition. For the wave, the pressure on the surface is equal to atmospheric pressure. Applying Equation (2.1.14) to the boundary surface $z = \xi(x, y, t)$, it follows that

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left((\nabla\Phi)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{p}{\rho} + g\xi = 0, \tag{2.2.1}$$

where $p = p_{atm} - T[\xi]$. Here p_{atm} denotes the atmospheric pressure and $T[\xi]$ the surface tension. Equation (2.2.1) can therefore be written as a function of atmospheric pressure and surface tension as follows

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left((\nabla\Phi)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + \frac{p_{atm}}{\rho} - \frac{T[\xi]}{\rho} + g\xi = 0 \quad . \tag{2.2.2}$$

For the case of constant atmospheric pressure and neglecting surface tension effects, we may set $p = p_{atm} = 0$ as the reference value at the free surface, and the equation becomes

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left((\nabla\Phi)^2 + \left(\frac{\partial \Phi}{\partial z}\right)^2\right) + g\xi = 0 \quad \text{at } z = \xi(x, y, t) \quad .$$

2.2.2 Kinematic boundary conditions on the surface. To simplify our study, we will limit the domain only to a plane formed by x and z. The direction of the flow remains the same; from left to right. Only the velocity components in the x and z directions, i.e u and w respectively, will be considered. Indeed the results obtained on the plane can be generalised throughout the space.

We know that the kinematic boundary conditions are related to the motion of water particles. It does not involve the action of forces (gravity, pressure, and more). It just describes the displacement of particles. The kinematic boundary condition states that any particles on the interface (surface of the wave) will remain on the interface and cannot leave the surface.

We know that a wave is simply a pattern formed by the motion of particles on a surface. It is a mixture of different displacements of particles. Note that the movement of the surface which is up and down describes the wave propagation due to the displacement of water particles. Indeed particles are in orbital movement around the equilibrium position, in a closed curve. The particular motion is circular in deep water whereas it is elliptic in shallow water (Holthuijsen, 2010).

As the surface of water changes, it can be defined as a function of position and time by $z - \xi(x, t) = 0$.

By applying the total derivative to the surface equation defined above, we obtained

$$\frac{dz}{dt} = \frac{d}{dt}(\xi(x, t)) = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}\frac{dx}{dt}$$

yielding

$$w = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}u. \tag{2.2.3}$$

This expression of the z component of velocity as a function of the surface elevation can be illustrated geometrically. Consider particle A on the surface of the wave as shown below in Figure 2.2.

The figure illustrate a surface, ξ, that moves in the z-direction. Along the z-axis the surface moves from point A in the direction of point D. However, the fluid particle at point A is moving in the direction of point E. The discussion provides a novel geometric interpretation of the mathematics of the kinematic boundary condition that states that fluid cannot cross the surface and that the velocity of the fluid normal to the surface has the same velocity as the surface.

- ⋄ τ is the tangential axis in the new reference frame.
- ⋄ n is the normal axis in the new reference frame, oriented outwards.
- ⋄ V_τ is the velocity component of the fluid particle along the tangential axis, τ.
- ⋄ V_n is the velocity component of the fluid particle along the normal axis, n.
- ⋄ C is the scalar component in the normal direction of the surface velocity.
- ⋄ ξ is the surface $z = z(\xi)$ elevation of the wave.

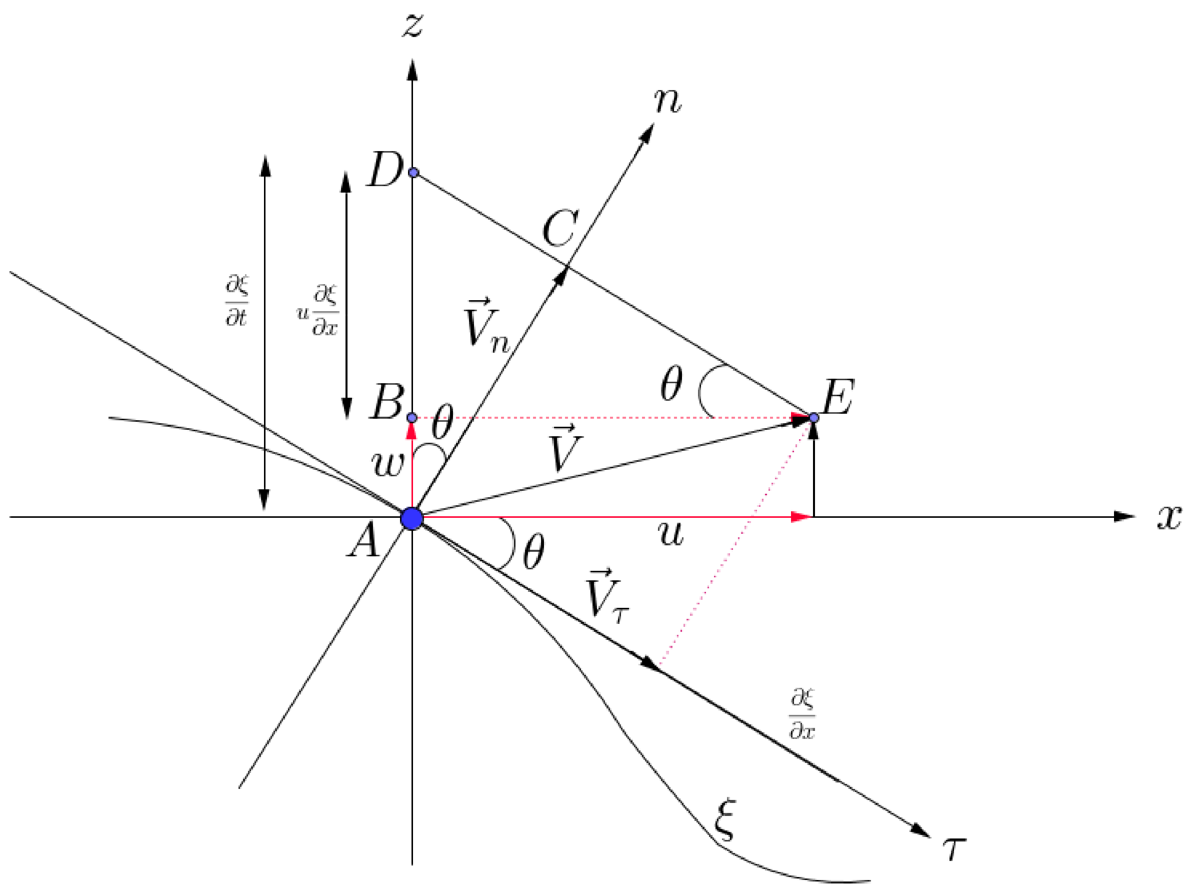


Figure 2.2: Particle and water surface in motion

So far, all our studies on the movement of waves and the displacement of particles have been done in Cartesian coordinates. We know that the displacement of the particle can also be defined in a new reference frame, defined by a normal and a tangential axis, and we defined θ as the angle between the z-axis of the Cartesian coordinate and the normal axis, n, of the new reference frame.

We can define a function F such that we have

$$F = z - \xi(x, t) = 0,$$

where $F(x, z, t)$ denotes the free surface. The vector ∇F is a normal vector to the surface and this gradient can be calculated as

$$\nabla F = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial z}\vec{k}\right)(z - \xi(x,t)) = \vec{k} - \frac{\partial \xi}{\partial x}\vec{i}.$$

The unit normal vector to the surface is obtained by the expression,

$$\vec{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{\vec{k} - \frac{\partial \xi}{\partial x}\vec{i}}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}.$$

We know that

$$\vec{V} = u\vec{i} + w\vec{k}.$$

The velocity of the fluid $\vec{V}$ can be written in the new reference frame also as follows

$$\vec{V} = V_n\vec{n} + V_\tau\vec{\tau}.$$

The dot product of $\vec{V}$ and $\vec{n}$ gives

$$V_n = \vec{V} \cdot \vec{n} = (u\vec{i} + w\vec{k}) \cdot \left(\frac{\vec{k} - (\frac{\partial \xi}{\partial x})\vec{i}}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\right),$$

yielding

$$V_n = \frac{1}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\left(-u\frac{\partial \xi}{\partial x} + w\right). \tag{2.2.4}$$

The velocity of the surface is denoted by $\frac{\partial \xi}{\partial t}\vec{k}$ and therefore

$$C = \frac{\partial \xi}{\partial t}\vec{k} \cdot \vec{n} = \frac{\partial \xi}{\partial t}\vec{k} \cdot \left(\frac{\vec{k} - (\frac{\partial \xi}{\partial x})\vec{i}}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\right),$$

yielding

$$C = \frac{1}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\frac{\partial \xi}{\partial t}. \tag{2.2.5}$$

The kinematic boundary condition states that the fluid particle cannot cross the surface, and therefore

$$V_n = C.$$

Equating the expressions of the normal velocity of the particle, Equation (2.2.4) to the normal velocity of the surface, Equation (2.2.5), it follows that

$$\frac{1}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\left(-u\frac{\partial \xi}{\partial x} + w\right) = \frac{1}{\sqrt{1^2 + (\frac{\partial \xi}{\partial x})^2}}\frac{\partial \xi}{\partial t}$$

yielding

$$w = \frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} \quad .$$

This corresponds to Equation 2.2.3. This relation is shown graphically in Figure 2.2. The magnitude of the surface velocity, $\frac{\partial \xi}{\partial t}$, consists of the sum of w and $-u\frac{\partial \xi}{\partial x}$. In the specific situation depicted, $\frac{\partial \xi}{\partial x} < 0$. By considering the triangle BED in Figure 2.2

$$\tan\theta = \frac{\frac{u\partial \xi}{\partial x}}{u} = \frac{\partial \xi}{\partial x} \quad (< 0). \tag{2.2.6}$$

Since θ is also the angle between the tangential and x-axis, $\tan\theta$ denotes the slope of the surface with respect to the x-direction which corresponds to Equation 2.2.6 obtained.

The kinematic boundary condition may be written in term of the velocity potential Φ, as follows

$$\frac{\partial \xi}{\partial t} = -\frac{\partial \Phi}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial \Phi}{\partial z} \qquad \text{at} \qquad z = \xi(x,t).$$

2.2.3 Kinematic boundary conditions at the seabed. At the seabed, we have $z = -h(x,t)$. By expanding the total derivative we have

$$\frac{dz}{dt} = -\frac{d}{dt}(h(x,t)) = -\frac{\partial h(x,t)}{\partial t} - \frac{\partial h(x,t)}{\partial x}\frac{dx}{dt}$$

yielding

$$w + \frac{\partial h(x,t)}{\partial t} + u\frac{\partial h(x,t)}{\partial x} = 0. \tag{2.2.7}$$

By substituting $u = \frac{\partial \Phi}{\partial x}$ and $w = \frac{\partial \Phi}{\partial z}$ into equation (2.2.7) we get

$$\frac{\partial \Phi}{\partial z} + \frac{\partial h(x,t)}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial h(x,t)}{\partial x} = 0.$$

In the case of a fixed bottom, we have $z = -h(x)$ and $\frac{\partial h(x)}{\partial t} = 0$. The condition at the seabed becomes

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial h(x)}{\partial x} = 0 \qquad \text{at} \qquad z = -h(x) \tag{2.2.8}$$

2.3 Boundary Value Problem and Linearisation

The governing equation and boundary conditions that describe the water wave with small amplitudes (Airy's wave) in our problem, written as a function of the velocity potential in the x and z-direction, is

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 & \text{at} \quad -h(x) \leq z \leq \xi(x,t) \\ \dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2}\left((\nabla \Phi)^2 + \left(\dfrac{\partial \Phi}{\partial z}\right)^2 \right) + g\xi = 0 & \text{at} \;\; z = \xi(x,t) \\ \dfrac{\partial \xi}{\partial t} + \nabla \Phi \cdot \nabla \xi = \dfrac{\partial \Phi}{\partial z} & \text{at} \quad z = \xi(x,t) \\ \dfrac{\partial \Phi}{\partial z} + \nabla \Phi \cdot \nabla h = 0 & \text{at} \quad z = -h(x). \end{cases}$$

The governing equation obtained is very difficult to solve, no complete solution is known for this equation, so we need to linearise and apply the boundary conditions. In our problem we have a linear wave, which means in our boundary value problem, we neglect the higher order terms, i.e the quadratic terms in Φ and z. The linearised governing equation and boundary conditions, (Dingemans, 1997), are therefore given by the following equations

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial x^2} + \dfrac{\partial^2 \Phi}{\partial z^2} = 0 & \text{at} \;\; -h(x) \leq z \leq \xi(x,t) \\ \dfrac{\partial \Phi}{\partial t} + g\xi = 0 & \text{at} \;\; z = \xi(x,t) \\ \dfrac{\partial \xi}{\partial t} = \dfrac{\partial \Phi}{\partial z} & \text{at} \quad z = \xi(x,t) \\ \dfrac{\partial \Phi}{\partial z} = 0 & \text{at} \quad z = -h(x). \end{cases} \tag{2.3.1}$$

The boundary value problem, Equation (2.3.1), can be solved by separation of variables yielding an expression for the surface elevation, ξ, which is a progressive wave without damping.

Finally, we note that the governing equation which describes the flow of water waves through porous media has been obtained at a microscopic level. Furthermore, appropriate boundary conditions have been defined. In the next chapter, we introduce a method with which to derive equations at a macroscopic level to model the average flow through a porous structure.

3. Porous Media and the Volume Averaging Method

In this section, we will describe the volume averaging method which is used to investigate transport in porous media. We will introduce the volume averaging method to seek a complete macroscopic governing equation of water waves. The volume average method will give the average properties that we need for our study.

3.1 Porous Media

The phenomena of transport in porous media are met in different disciplines, from civil engineering with transport and diffusion of pollutants in aquifers, to reservoir engineering with the flow of oil, water and gas, also in oceanography.

The porous medium is defined as a medium containing pores. A skeletal portion of the porous medium is called the matrix or frame. The pores are typically filled with fluid. The porous medium is characterised by different parameters of which some will be introduce below (Lyons et al., 2015).

3.1.1 Characteristics of porous media. The flow in porous media differs depending on the material in which it takes place. Different parameters can characterise the material. In this study, only porosity and permeability are presented.

- ◇ Porosity: the porosity of a material is a measure of the void spaces in a material. It is the fraction of the volume of voids over the total volume. It is a unitless number between 0 and 1, and also can be expressed as a percentage.
- ◇ Permeability: the permeability of a material is a measure of the ability to transmit fluids.

3.1.2 Examples of porous media. The parameters defined above characterise porous media. There are different types of porous media, depending on their structure and connectivity, for example cement, sand, fissure rock, soil, dikes, breakwaters, and seawalls are all porous media. In this study, we will only consider porous media such as those used in coastal engineering and oceanography.

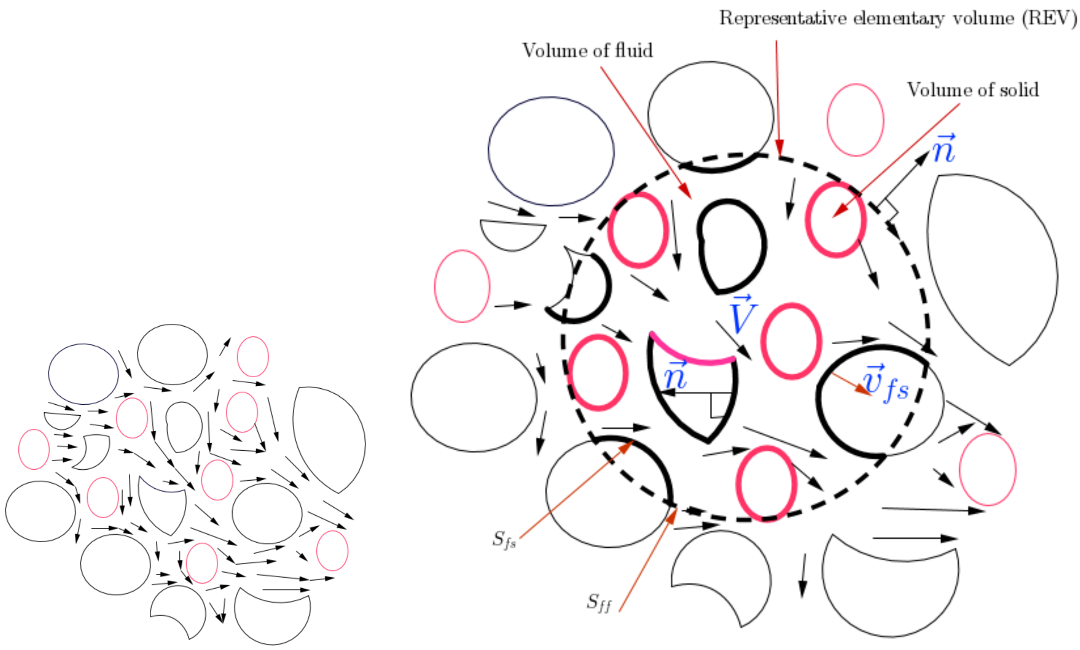
3.2 Volume Averaging Method

The volume averaging method starts by defining a representative elementary volume (averaging volume) of the porous medium. The averaging volume is representative of the fluids and solids present. Depending on the type of problem, there are three different types of averaging methods, namely the Eulerian averaging method, the Lagrangian averaging method and the Molecular statistical averaging method (Faghri and Zhang, 2006). All throughout our study we will use the Eulerian averaging method.

In volume averaging methods, the volume of solid and fluid can be functions of position and time.

Let $\psi(x, y, z, t)$ be any tensor function we need to know such as velocity, pressure, and more.

The figure below presents the processes of applying the Eulerian volume averaging method in a porous medium.



(a) Flow in porous media.

(b) Control volume inside .

Figure 3.1: Process of volume averaging method.

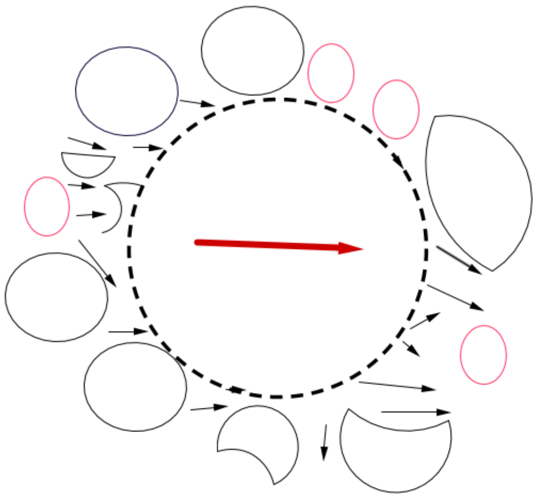


Figure 3.2: Average obtained

Figure 3.1(a) presents the porous medium as a continuous medium with properties that we need to determine. The figure presents the flow of fluid inside the medium. That means we have different phases. This is also called the original flow in the porous medium.

The second picture, Figure 3.1(b), presents the control volume extract from the original flow, in order to apply a volume averaging method and determine all properties needed. The control volume is also called the representative elementary volume (REV). All calculations will be done inside the REV.

Figure 3.2 presents the averaging parameter or property obtained by using the volume averaging method. This property can be the velocity, pressure or some other parameters, but Figure 3.2 illustrates the average of a vector quantity such as velocity.

We defined $\langle\psi\rangle$ as the volume averaging operator of ψ such that

$$\langle\psi\rangle = \frac{1}{V_a}\iiint_{V_f}\psi dV,$$

where V_f is the volume of fluid in the REV and V_a is the total volume of the REV. The volume average is also called the local volume average or extrinsic phase average. We define the intrinsic phase average, also called the pore average, as

$$\langle\psi\rangle^f = \frac{1}{V_f}\iiint_{V_f}\psi dV.$$

Notice that the intrinsic phase average and the extrinsic phase average are related by the relation

$$\langle\psi\rangle = \epsilon\langle\psi\rangle^f,$$

where ϵ represents the porosity.

3.2.1 Properties of the volume averaging method. In this section, we will present some important properties of the volume averaging method. Suppose we have two fluid phase tensor functions ψ and λ with similar order and α is a constant. The following properties then apply:

$$\begin{aligned} \langle 1\rangle^f &= 1, & \langle 1\rangle &= \epsilon,\\ \langle\psi+\lambda\rangle &= \langle\psi\rangle+\langle\lambda\rangle, & \langle\alpha\psi\rangle &= \alpha\langle\psi\rangle,\\ \langle\langle\psi\rangle\rangle &= \epsilon\langle\psi\rangle, & \langle\psi\lambda\rangle^f &= \langle\psi\rangle^f\langle\lambda\rangle^f+\{\psi\lambda\}^f,\\ \{\psi\} &= \psi-\langle\psi\rangle_f, \end{aligned}$$

where $\{\psi\}$ is the deviation relative to the intrinsic phase average.

3.2.2 Spatial averaging theorem and Slattery's averaging theorem. The spatial averaging theorem expresses the gradient of an average of some tensorial function as the average of the gradient of the function. An intermediate step towards the averaging theorem is Slattery's averaging theorem which may be expressed as

$$\nabla_t\iiint_{V_f}\psi dV = \iint_{S_{ff}}\vec{n}\psi dS.$$

S_{ff} is the fluid-fluid interface in the REV, and dS represents the element of the surface.

This equation is called *Slattery's Averaging Theorem* (Whitaker, 1998). It is valid for any representative element of volume which is constant in magnitude and shape.

3.2.3 Gradient and divergence theorem. The averaging theorem presented above is expressed in terms of the surface element at the fluid-fluid interface. In reality, S_{ff} is difficult to quantify. An easier way to apply the averaging method is to use the divergence theorem in a REV. By applying the method on the fluid-solid interface (S_{fs}), we obtain two important results (Whitaker, 1998).

$$\langle\nabla_t\psi\rangle = \nabla_t\langle\psi\rangle + \frac{1}{V_a}\iint_{S_{fs}}\vec{n}\psi dS,$$

and

$$\langle \nabla_t \cdot \vec{\psi} \rangle = \nabla_t \cdot \langle \vec{\psi} \rangle + \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \vec{\psi} dS.$$

These equations can also be expressed as functions of the intrinsic phase average such that

$$\langle \nabla_t \psi \rangle = \epsilon \nabla_t \langle \psi \rangle^f + \frac{1}{V_a} \iint_{S_{fs}} \vec{n}\{\psi\} dS,$$

and

$$\langle \nabla_t \cdot \vec{\psi} \rangle = \epsilon \nabla_t \cdot \langle \vec{\psi} \rangle^f + \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \{\vec{\psi}\} dS,$$

with $\{\psi\} = \psi - \langle \psi \rangle_f$. Furthermore

$$\langle \nabla_t^2 \psi \rangle = \nabla_t^2 \langle \psi \rangle + \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \nabla_t \psi dS + \frac{1}{V_a} \nabla_t \cdot \left[\iint_{S_{fs}} \vec{n} \psi dS \right].$$

Also, we have the porosity expressed as follows :

$$\nabla_t \epsilon = -\frac{1}{V_a} \iint_{S_{fs}} \vec{n} dS, \qquad \nabla_t \epsilon = \frac{1}{V_a} \iint_{S_{ff}} \vec{n} dS.$$

To allow physical properties to change with respect to time within the REV, we need to determine the change of a tensorial function with respect to time. This change can be obtained as follows:

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle = \frac{\partial \langle \psi \rangle}{\partial t} - \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \vec{v}_{fs} \psi dS,$$

where $\vec{v}_{fs}$ is the velocity of the fluid-solid surface.

We can write this equation as a function of the intrinsic phase averaging as follows

$$\epsilon \left\langle \frac{\partial \psi}{\partial t} \right\rangle^f = \frac{\partial}{\partial t} \left(\epsilon \langle \psi \rangle_f \right) - \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \vec{v}_{fs} \psi dS.$$

By taking, $\psi = 1$ we obtain the equation below

$$\frac{\partial \epsilon}{\partial t} = \frac{1}{V_a} \iint_{S_{fs}} \vec{n} \cdot \vec{v}_{fs} dS.$$

Now that we explained the volume averaging method, we need to apply this method to the governing equation obtained in Chapter 2. This will be the focus of the next chapter.

4. Volume Averaging of the Water Wave Equations

In all of our previous sections, we have determined the governing equation and boundary conditions which describes the movement of water waves with small amplitudes by using Airy's wave theory. Then we defined porous media and introduced the volume averaging method to determine macroscopic parameters in porous media. In this section, we will apply the volume averaging method to our governing equation and boundary conditions obtained.

4.1 Volume Averaging of Laplace Equation

Let us first apply the volume averaging method to the governing equation of the boundary value problem, Equation (2.3.1), obtained in Chapter 2 which is Laplace's equation.

The volume averaging method gives the following equation

$$\Big\langle 0 \Big\rangle = \Big\langle \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} \Big\rangle.$$

The average of the sum of two functions is equal to the sum of the averages of the functions. The relation now becomes

$$0 = \Big\langle \frac{\partial^2 \Phi}{\partial x^2} \Big\rangle + \Big\langle \frac{\partial^2 \Phi}{\partial z^2} \Big\rangle.$$

We know that all of the averaged terms obtained above are second derivatives of the function Φ with respect to x and z. By applying the divergence theorem defined in Chapter 3, we get

$$0 = \frac{\partial}{\partial x}\Big\langle \frac{\partial \Phi}{\partial x} \Big\rangle + \frac{\partial}{\partial z}\Big\langle \frac{\partial \Phi}{\partial z} \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} \left[n_x \frac{\partial \Phi}{\partial x} + n_z \frac{\partial \Phi}{\partial z} \right] dS, \tag{4.1.1}$$

where $\vec{n} = n_x \vec{i} + n_z \vec{k}$.

The gradient theorem applied to Φ in x and z-direction gives

$$\Big\langle \frac{\partial \Phi}{\partial x} \Big\rangle = \frac{\partial}{\partial x}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} n_x \Phi dS, \tag{4.1.2}$$

and

$$\Big\langle \frac{\partial \Phi}{\partial z} \Big\rangle = \frac{\partial}{\partial z}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} n_z \Phi dS, \tag{4.1.3}$$

respectively. By substituting (4.1.2) and (4.1.3) into (4.1.1) it follows

$$0 = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} n_x \Phi dS \right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} n_z \Phi dS \right] + \frac{1}{V_a}\iint_{S_{fs}} \left[n_x \frac{\partial \Phi}{\partial x} + n_z \frac{\partial \Phi}{\partial z} \right] dS$$

yielding

$$0 = \frac{\partial^2}{\partial x^2}\Big\langle \Phi \Big\rangle + \frac{\partial^2}{\partial z^2}\Big\langle \Phi \Big\rangle + \frac{\partial}{\partial x}\left[\frac{1}{V_a}\iint_{S_{fs}} n_x \Phi dS \right] + \frac{\partial}{\partial z}\left[\frac{1}{V_a}\iint_{S_{fs}} n_z \Phi dS \right] + \frac{1}{V_a}\iint_{S_{fs}} \left[n_x \frac{\partial \Phi}{\partial x} + n_z \frac{\partial \Phi}{\partial z} \right] dS.$$

4.2 Volume Averaging of the Boundary Conditions

Now we need to apply the volume averaging method to the boundary conditions of our Equation (2.3.1). The method is applied for each equation as follows.

4.2.1 Volume average for the dynamic boundary condition. By applying the volume averaging method to the dynamic boundary condition obtained, we have

$$\Big\langle \frac{\partial \Phi}{\partial t} + g\xi \Big\rangle = \Big\langle 0 \Big\rangle.$$

The property of the average of a sum gives

$$\Big\langle \frac{\partial \Phi}{\partial t} \Big\rangle + \Big\langle g\xi \Big\rangle = 0. \tag{4.2.1}$$

The gravitational acceleration is known and constant. The second term of the left side becomes

$$\Big\langle g\xi \Big\rangle = g\Big\langle \xi \Big\rangle. \tag{4.2.2}$$

By using the definition presented in Chapter 3, the first term in Equation (4.2.1) which is the average in time of the velocity potential becomes

$$\Big\langle \frac{\partial \Phi}{\partial t} \Big\rangle = \frac{\partial}{\partial t}\Big\langle \Phi \Big\rangle - \frac{1}{V_a}\iint_{S_{fs}} \Phi \vec{n} \cdot \vec{v}_{fs} dS. \tag{4.2.3}$$

Substituting Equations (4.2.3) and (4.2.2) into (4.2.1), we obtained the following equation

$$\frac{\partial}{\partial t}\Big\langle \Phi \Big\rangle + g\Big\langle \xi \Big\rangle = \frac{1}{V_a}\iint_{S_{fs}} \Phi \vec{n} \cdot \vec{v}_{fs} dS.$$

4.2.2 Volume average for the kinematic boundary conditions. The volume averaging method applied to the kinematic boundary conditions for the governing equation gives

$$\Big\langle \frac{\partial \xi}{\partial t} \Big\rangle = \Big\langle \frac{\partial \Phi}{\partial z} \Big\rangle$$
$$\frac{\partial}{\partial t}\Big\langle \xi \Big\rangle - \frac{1}{V_a}\iint_{S_{fs}} \xi \vec{n} \cdot \vec{v}_{fs} dS = \frac{\partial}{\partial z}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} \Phi n_z dS,$$

yielding

$$\frac{\partial}{\partial t}\Big\langle \xi \Big\rangle = \frac{\partial}{\partial z}\Big\langle \Phi \Big\rangle + \frac{1}{V_a}\iint_{S_{fs}} \vec{n} \cdot \left[\xi \vec{v}_{fs} + \Phi \vec{k}\right] dS \qquad \text{at } z = \xi.$$

Applying the volume averaging method to the kinematic boundary condition at the seabed we obtain

$$\Big\langle \frac{\partial \Phi}{\partial z} \Big\rangle = 0.$$

Therefore

$$\frac{\partial}{\partial z}\Big\langle \Phi \Big\rangle = -\frac{1}{V_a}\iint_{S_{fs}} n_z \Phi dS \qquad \text{at } \ z = -h.$$

4.3 Volume Averaged Governing Equations

The macroscopic volume averaging equation obtained from the Equation (2.3.1) is

$$\begin{cases} \dfrac{\partial^2}{\partial x^2}\left\langle \Phi \right\rangle + \dfrac{\partial^2}{\partial z^2}\left\langle \Phi \right\rangle + \mathbf{H} = 0, & \text{(4.3.1a)} \\ \dfrac{\partial}{\partial t}\left\langle \Phi \right\rangle + g\left\langle \xi \right\rangle = \dfrac{1}{V_a}\displaystyle\iint_{S_{fs}} \Phi\vec{n}\cdot\vec{v}_{fs} dS & \text{at } z = \left\langle \xi \right\rangle, \quad \text{(4.3.1b)} \\ \dfrac{\partial}{\partial t}\left\langle \xi \right\rangle = \dfrac{\partial}{\partial z}\left\langle \Phi \right\rangle + \dfrac{1}{V_a}\displaystyle\iint_{S_{fs}} \vec{n}\cdot\left[\xi\vec{v}_{fs} + \Phi\vec{k}\right] dS & \text{at } z = \left\langle \xi \right\rangle, \quad \text{(4.3.1c)} \\ \dfrac{\partial}{\partial z}\left\langle \Phi \right\rangle = -\dfrac{1}{V_a}\displaystyle\iint_{S_{fs}} n_z \Phi dS & \text{at } z = -h, \quad \text{(4.3.1d)} \end{cases}$$

where

$$\mathbf{H} = \frac{\partial}{\partial x}\left[\frac{1}{V_a}\iint_{S_{fs}} n_x\Phi dS\right] + \frac{\partial}{\partial z}\left[\frac{1}{V_a}\iint_{S_{fs}} n_z\Phi dS\right] + \frac{1}{V_a}\iint_{S_{fs}}\left[n_x\frac{\partial\Phi}{\partial x} + n_z\frac{\partial\Phi}{\partial z}\right] dS.$$

4.4 Illustration of Closure Modelling

In this study, our averaged Equations (4.3.1a) to (4.3.1d) have integral terms and to solve them we need to know the area of the interface, S_{fs}, contained within V_a. Also, we need to know the velocity potential, Φ, at these interfaces. Therefore, these equations become difficult to solve. To make the equations useful, we must assume some shape for the area at the interface and express the integral in terms of $\left\langle \Phi \right\rangle$ and ϵ, which are macroscopic parameters. This process is called closure modelling. Only after closure modelling, can the equations be solved analytically or numerically.

For closure modelling we need information about the flow on the microscopic level. For this we will consider the experimental microscopic flow measured by (Terblanche et al., 2018). This study of the interactions of water waves in porous media has been conducted in the laboratory of the Council for Scientific and Industrial Research (CSIR) at Stellenbosch University, South Africa. The experiment consisted of constructing from timber beams a rectangular porous structure arranged in a non-staggered array. The structure interacted with a monochromatic wave in a glass wave flume which was 1 m deep, 33 m long and 0.7 m wide. Shallow water waves were generated by a piston. The monochromatic waves generated had an incident height of 0.032 m and a period of 1 s. The region measured was illuminated with a light source, and filmed with a high-speed camera at 100 frames per second. The incident and reflected wave height, and the wave transmission through the structure were measured. Note the flow field was visualized with fluorescent green polyethylene microspheres.

Figure 4.1: Glass flume at the CSIR Coastal and Hydraulics (Terblanche et al., 2018)

Figure 4.2: Image of the experimental (Terblanche et al., 2018)

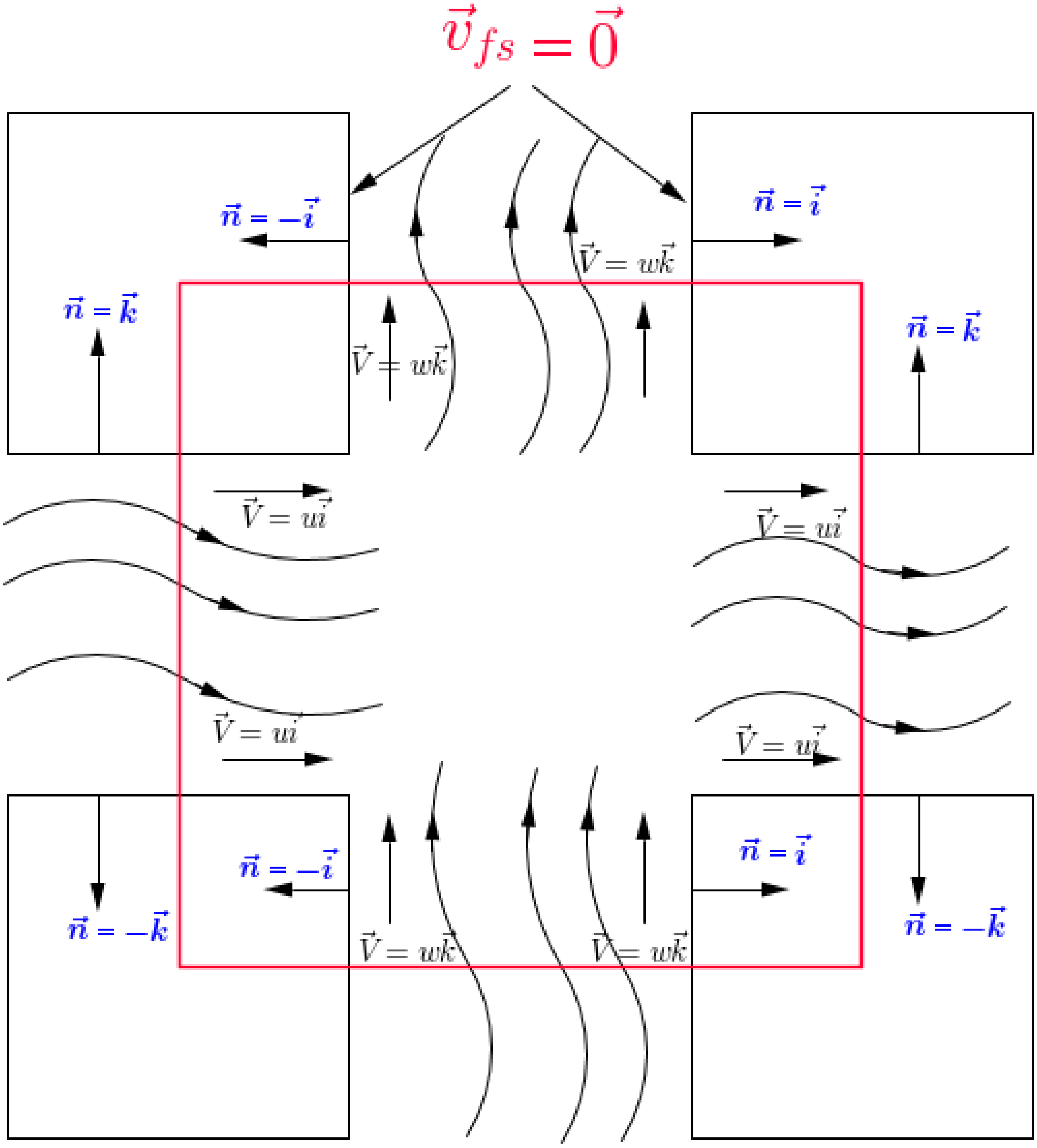


Figure 4.3: Macroscopic geometry and velocity components

The velocity components have been drawn in positive directions. For the measurement, the solid was stationary which implies that $\vec{v}_{fs}$=0. The flow is mostly parallel to the fluid-solid interfaces.

Therefore, in the vertical channels with n_x components, the microscopic fluid velocity at the fluid-solid interface only has z-components. This implies that the third term in Equation (4.3.1d) and the third term in Equation (4.3.1c) are zero

$$\iint_{S_{fs}} \left[n_x \frac{\partial \Phi}{\partial x} + n_z \frac{\partial \Phi}{\partial z} \right] dS = 0,$$

since $\frac{\partial \Phi}{\partial x} = 0$ and $n_z = 0$.

Similarly, in the horizontal channels, at the fluid-solid interfaces, the microscopic fluid velocity only has x-components, which implies that

$$\iint_{S_{fs}} \left[n_x \frac{\partial \Phi}{\partial x} + n_z \frac{\partial \Phi}{\partial z} \right] dS = 0,$$

since $\frac{\partial \Phi}{\partial z} = 0$ and $n_x = 0$.

The macroscopic governing equation becomes

$$\begin{cases} \frac{\partial^2}{\partial x^2}\left\langle \Phi \right\rangle + \frac{\partial^2}{\partial z^2}\left\langle \Phi \right\rangle + \mathbf{H}^* = 0, & \text{(4.4.1a)} \\ \frac{\partial}{\partial t}\left\langle \Phi \right\rangle + g\left\langle \xi \right\rangle = 0, & \text{(4.4.1b)} \\ \frac{\partial}{\partial t}\left\langle \xi \right\rangle = \frac{\partial}{\partial z}\left\langle \Phi \right\rangle + \frac{1}{V_a} \iint_{S_{fs}} n_z \Phi dS, & \text{(4.4.1c)} \\ \frac{\partial}{\partial z}\left\langle \Phi \right\rangle = -\frac{1}{V_a} \iint_{S_{fs}} n_z \Phi dS, & \text{(4.4.1d)} \end{cases}$$

where

$$\mathbf{H}^* = \frac{\partial}{\partial x}\left[\frac{1}{V_a} \iint_{S_{fs}} n_x \Phi dS \right] + \frac{\partial}{\partial z}\left[\frac{1}{V_a} \iint_{S_{fs}} n_z \Phi dS \right].$$

This approach must also be applied to the other surface integrals in set above, but this is beyond the scope of this study.

5. Conclusion and Future work

5.1 Conclusion

The phenomenon of water flow through porous media continue to fascinate a wide range of researchers, scientists, and engineers due to its importance in many fields of study and industrial applications. The need to understand these interactions has caused a rapid expansion of research in different areas these past decades and continues to produce a huge amount of theoretical models work and experimentation. Worldwide, different groups of researchers and laboratories work hard to better understand these phenomena.

A novel approach to illustrate the kinematic boundary conditions was put forward in this study. Volume averaging of the microscopic boundary value problem is not presented in literature and is a unique approach attempted. In this study, we described the equation that governs wave motion in the absence of porous media by using Airy's wave theory. Then we presented the set of equations that defined linear wave motion. We used volume averaging theory to obtain the set of the macroscopic equations governing linear water waves in porous media. Finally, an attempt was made to do a closure modelling on the governing macroscopic equation.

To be able to get a complete, closed macroscopic boundary value problem a study must be made of potential theory within the porous channels. Only when we have a closed macroscopic boundary value problem can we solve for $\langle \xi \rangle$ and $\langle \Phi \rangle$.

5.2 Future Work

For further research, a complete close macroscopic boundary value problem must be made by studying potential theory. The model must be validated by comparing it with experiments made in the laboratory. Airy's wave theory used for this study is not the only theory that could be used. Stokes wave and Boussinesq wave could also be used.

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